

Algebraic cycles on smooth varieties fibred by varieties with small Chow groups

Charles Vial

Abstract

We study the Chow groups of smooth complex varieties X which are fibred by varieties with small Chow groups, e.g. varieties fibred by low degree complete intersections or cellular varieties. As an application, we give new examples of smooth projective varieties satisfying conjectures on algebraic cycles such as the Hodge conjecture, the Lefschetz standard conjecture, Kimura's finite-dimensionality conjecture and Murre's conjectures. If X is fibred by cellular varieties over a curve, then we show that X is Kimura finite-dimensional. If X is fibred by cubic fivefolds over a curve, then we show that X satisfies Murre's conjectures as well as the standard conjectures.

Introduction

Let k be a field and let Ω be a universal domain over k , that is Ω is an algebraically closed field of infinite transcendence degree over k . For X a variety over k , the group $CH_i(X)$ denotes the rational Chow group of i -cycles on X modulo rational equivalence.

Let $f : X \rightarrow S$ be a family defined over k of projective varieties over a quasi-projective variety S . We ask: What can be said of the Chow groups of X in terms of the Chow groups of S and of the Chow groups of the fibres? For instance, if $f : X \rightarrow S$ is a projective bundle over a smooth quasi-projective variety S , then the projective bundle formula gives, for all l , an isomorphism induced by an action of correspondences $\bigoplus_{0 \leq j \leq \dim X - \dim S} CH_{l-j}(S) \xrightarrow{\sim} CH_l(X)$.

Another example is the following. If X is smooth and if a general fibre F of $f : X \rightarrow S$ satisfies $CH_0(F) = \mathbf{Q}$, then there exists a closed subscheme $\tilde{S} \hookrightarrow X$ of dimension $\dim S$ such that the pushforward map $CH_0(\tilde{S}) \rightarrow CH_0(X)$ is surjective. Indeed, let $\tilde{S} \hookrightarrow X$ be a smooth linear section of X of dimension $\dim S$ which dominates S and let U be a dense open subset of S such that $CH_0(X_u) = \mathbf{Q}$ for all $u \in U$. Consider then a zero-cycle α in X . By Chow's moving lemma, α is rationally equivalent to a zero-cycle $\beta = \sum a_i [p_i]$ supported on $X|_U$, that is such that p_i is a closed point of $X|_U$ for all i . But then, each p_i is rationally equivalent to a rational multiple of a closed point of \tilde{S} . Therefore, α is rationally equivalent to a cycle supported on \tilde{S} . In other words, $CH_0(X)$ is supported in dimension $\dim S$, or we say that $CH_0(X)$ has niveau $\dim S$. A more precise statement of this observation is theorem 2.12. It can also be proved that if a general fibre F of $f : X \rightarrow S$ is such that $CH_0(F_\Omega)$ has niveau ≤ 1 , then $CH_0(X)$ has niveau $\leq \dim S + 1$.

More generally, Laterveer [11] defines a notion of niveau on Chow groups as follows. For a variety X , the group $CH_i(X)$ is said to have *niveau* $\leq n$ if there exists a closed subscheme Z of X of dimension $\leq i + n$ such that $CH_i(Z) \rightarrow CH_i(X)$ is surjective, in other words if the i -cycles on X are supported in dimension $i + n$.

In that context, a somewhat more precise question is: What can be said about the niveau of the Chow groups of X in terms of the niveau of the Chow groups of the fibres of $f : X \rightarrow S$?

The general philosophy here is the following : If X is a smooth variety which is fibred by varieties with small Chow groups over a base B of small dimension, then X has small Chow groups. Somewhat more precisely, if $CH_*(X_b)$ has niveau $\leq n$ for all complex point $b \in B$, then we expect $CH_*(X)$ to have niveau $\leq n + \dim B$. We cannot prove such a general statement but we wish to tackle that question when some of the Chow groups of the fibres of f have niveau 0. The most general statements we obtain are theorems 2.5, 2.9 and 2.11. When X is defined over the complex numbers, those theorems become the following.

Theorem 1. [Theorem 3.4] *Let $f : X \rightarrow B$ be a complex projective dominant morphism onto a complex quasi-projective variety B of dimension d_B . Assume that $CH_i(X_b) = \mathbf{Q}$ for all $i \leq l$ and all closed point $b \in B$. Then $CH_i(X)$ has niveau $\leq d_B$ for all $i \leq l$.*

Theorem 2. [Theorem 3.5] *Let $f : X \rightarrow C$ be a complex generically smooth projective dominant morphism onto a smooth complex curve. Assume that $CH_i(X_c)$ is finitely generated for all closed points $c \in C$ and all $i \leq l$, then $CH_i(X)$ has niveau ≤ 1 for all $i \leq l$.*

Theorem 3. [Theorem 3.6] *Let $f : X \rightarrow B$ be a complex generically smooth projective dominant morphism onto a smooth quasi-projective complex variety B of dimension d_B . Assume that the singular locus of f in B is finite. Assume also that*

- $CH_i(X_b) = \mathbf{Q}$ for all closed points $b \in B$ and all $i < l$,
- $CH_l(X_b)$ is finitely generated for all closed points $b \in B$.

Then $CH_i(X)$ has niveau $\leq d_B$ for all $i \leq l$.

We are then interested in using those results to give new examples of smooth projective varieties satisfying some of the conjectures on algebraic cycles.

Let now X be a smooth projective complex variety of dimension d_X . We write $H_i(X)$ for the rational homology group $H_i(X, \mathbf{Q})$, this group is isomorphic to $H^i(X, \mathbf{Q})^\vee$. There is a cycle class map $cl_i : CH_i(X) \rightarrow H_{2i}(X)$. The Hodge conjecture describes the image of cl_i as being the subset of $H_{2i}(X)$ consisting of Hodge classes. In other words, the Hodge conjecture describes explicitly which of the transcendental cycles in $H_{2i}(X)$ are algebraic in terms of the Hodge structure of $H_{2i}(X)$. In particular the Hodge conjecture stipulates the existence for all $i \leq d_X$ of cycles $L \in CH^i(X \times X)$ that induce isomorphisms $L_* : H_i(X) \rightarrow H_{2d_X-i}(X)$. This is Grothendieck's Lefschetz standard conjecture for X . It implies yet another conjecture on the existence of algebraic cycles, namely the Künneth standard conjecture which asserts the existence of cycles

$\pi_i \in CH_{d_X}(X \times X)$ that induce the Künneth projectors $(\pi_i)_* : H_*(X) \rightarrow H_i(X) \rightarrow H_*(X)$. Understanding the kernel of the cycle class map is another matter. It was conjectured by Bloch and Beilinson that there should exist a descending filtration F on Chow groups of smooth projective varieties which is functorial with respect to the action of correspondences and which satisfies $F^1 CH_i(X) = \text{Ker}(CH_i(X) \rightarrow H_{2i}(X))$. It turns out that the existence of such a filtration on Chow groups is related to the notion of finite-dimensionality as introduced by Kimura [10] and independently by O'Sullivan. These authors have conjectured that every smooth projective variety should be finite-dimensional. A consequence of finite-dimensionality for X is that the kernel of the cycle class map $cl : CH_{d_X}(X \times X) \rightarrow H_{2d_X}(X \times X)$ is nilpotent for the composition law on correspondences. This in turn implies that idempotents in the image of cl can be lifted to idempotents in $CH_{d_X}(X \times X)$. We thus see that the finite-dimensionality condition on X gives the existence of algebraic cycles which are idempotents modulo rational equivalence. In particular if the Künneth projectors in homology are algebraic, then they lift to idempotents adding to the identity modulo rational equivalence. In [12], Murre conjectured the following.

- (A) X has a Chow-Künneth decomposition $\{\pi_0, \dots, \pi_{2d}\}$: There exist mutually orthogonal idempotents $\pi_0, \dots, \pi_{2d} \in CH_{d_X}(X \times X)$ adding to the identity such that $(\pi_i)_* H_*(X) = H_i(X)$ for all i .
- (B) $\pi_0, \dots, \pi_{2l-1}, \pi_{d+l+1}, \dots, \pi_{2d}$ act trivially on $CH_l(X)$ for all l .
- (C) $F^i CH_l(X) := \text{Ker}(\pi_{2l}) \cap \dots \cap \text{Ker}(\pi_{2l+i-1})$ doesn't depend on the choice of the π_j 's. Here the π_j 's are acting on $CH_l(X)$.
- (D) $F^1 CH_l(X) = CH_l(X)_{\text{hom}}$.

A variety X that satisfies conjectures (A), (B) and (D) is said to have a *Murre decomposition*. Jannsen proved [9] that Murre's conjectures are true for all smooth projective varieties if and only if Bloch and Beilinson's conjecture is true for all smooth projective varieties.

The Lefschetz standard conjecture for X can be reformulated as saying that X satisfies the Künneth standard conjecture and that the morphisms of motives modulo homological equivalence $(X, \pi_{2d_X-i}^{\text{hom}}) \rightarrow (X, \pi_i^{\text{hom}}, d_X - i)$ induced by intersecting $d_X - i$ times with a hyperplane section are isomorphisms for all $0 \leq i \leq d_X$. If X is assumed to be finite-dimensional in the sense of Kimura, this isomorphism lifts to rational equivalence to give an isomorphism $(X, \pi_{2d_X-i}) \rightarrow (X, \pi_i, d_X - i)$ of Chow motives. This is referred to as the *motivic Lefschetz conjecture*.

It has been known since the work of Bloch and Srinivas [2] that if X has a small Chow group of zero-cycles, then this has consequences for the support of the cohomology of X and for codimension-2 cycles on X . This was generalised by Laterveer [11], Paranjape [14] and Schoen [15] among others. They proved that for X to have small Chow groups has implications on the motive of X . The notion of "small" that we use here for Chow groups is related to the notion of niveau. For example, Laterveer proves that if the Chow groups of X have niveau ≤ 2 then homological and algebraic equivalence agree on X and if they have niveau ≤ 3 then the Hodge conjecture holds for X . In [20], we proved

that if the Chow groups of X have niveau ≤ 1 then the motive of X is finite-dimensional in the sense of Kimura [10]. In the first section we review those results and we give a complement by proving that if the Chow groups of X have niveau ≤ 2 then X satisfies the Lefschetz standard conjecture, see theorem 1.5.

As a byproduct of theorems 1, 2 and 3, we are able to exhibit new examples of varieties for which we can prove some of the conjectures on algebraic cycles. This is the object of section 4 and as samples we prove the following results which are concerned with smooth projective varieties fibred either by cellular varieties, or hypersurfaces of very low degree.

Theorem 4 (Theorem 4.8). *Let $f : X \rightarrow B$ be a dominant morphism between smooth projective complex varieties whose closed fibres are cellular varieties. Then,*

- *if $\dim B \leq 1$, X is finite-dimensional in the sense of Kimura and X satisfies Murre's conjectures as well as the motivic Lefschetz conjecture ;*

Assume now that f is connected and that there exists a finite set of closed points Σ in B such that f is smooth over $B - \Sigma$. Then,

- *if $\dim B \leq 2$ and $\dim X \leq 6$, X satisfies Grothendieck's standard conjectures ;*
- *if $\dim B \leq 3$ and $\dim X \leq 7$, X satisfies the Hodge conjecture.*

The assumption on the singular locus of f being finite is the same as the one appearing in [5] where the construction of relative Chow-Künneth decompositions is considered. However, here we do not require the restriction of $f : X \rightarrow B$ to $B - \Sigma$ to be a family of relative cellular varieties, nor do we require it to be locally trivial, as is for example the case in [8] or [7].

The next theorem follows from theorem 2 together with the results on Chow-Künneth decompositions that appear in [20] and those on Murre's conjectures that appear in [18], all of which are recalled in section 1.

Theorem 5 (Theorem 4.5). *Let X be a smooth projective complex variety fibred by cubic fivefolds over a curve. Then X satisfies Murre's conjectures as well as the motivic Lefschetz conjecture.*

Let Q be a quadric hypersurface. Then it is known that $CH_i(Q) = \mathbf{Q}$ for all $i < \frac{\dim Q}{2}$ so that theorem 1 together with the results of section 1 gives

Theorem 6 (Theorem 4.2). *Let X be a smooth projective complex variety fibred by quadric hypersurfaces over a smooth projective variety B . Then*

- *if $\dim B \leq 1$, X is finite-dimensional in the sense of Kimura ;*
- *if $\dim B \leq 2$, X satisfies Grothendieck's standard conjectures ;*
- *if $\dim B \leq 3$, X satisfies the Hodge conjecture.*

It should be noted that, in the above theorem, no assumption on the singular locus of the quadric fibration $X \rightarrow B$ is made.

Notations. Chow groups are always meant with rational coefficients. The group $CH_i(X)$ is the \mathbf{Q} -vectorspace with basis the i -dimensional irreducible subschemes of X modulo rational equivalence. It is said to be *finitely generated* if it is finitely generated as a \mathbf{Q} -vectorspace, i.e. if it is a finite-dimensional \mathbf{Q} -vector space.

Acknowledgements. Thanks to Mingmin Shen for a discussion leading to the proof of proposition 2.6. Thanks to Burt Totaro for suggesting the statement of lemma 3.1. This work is supported by a Thomas Nevile Research Fellowship at Magdalene College, Cambridge and an EPSRC Postdoctoral Fellowship under grant EP/H028870/1. I would like to thank both institutions for their support.

1 Varieties with small Chow groups

In this section, except for theorem 1.5, we review known results about varieties with small Chow groups. Varieties are defined over a field k of characteristic zero and Ω denotes a universal domain over k .

The following definition is taken from Laterveer [11].

Definition. Let X be a variety. The Chow group $CH_i(X)$ of i -cycles on X modulo rational equivalence is said to have *niveau* $\leq r$ if there exists a closed subscheme $Y \subset X$ of dimension $i+r$ such that the proper pushforward map $CH_i(Y) \rightarrow CH_i(X)$ is surjective.

Definition. Let X be a smooth projective variety. The Chow group $CH_i(X)$ of i -cycles on X modulo rational equivalence is said to be *representable* if there exists a smooth projective curve C and a correspondence $\Gamma \in CH_{i+1}(C \times X)$ such that the induced map $\Gamma_* : CH_0(C) \rightarrow CH_i(X)$ is surjective.

Lemma 1.1. *If X is smooth projective, the Chow group $CH_i(X)$ has niveau ≤ 1 if and only if it is representable.*

Proof. Let \tilde{Y} be a desingularisation (or an alteration in positive characteristic) of a closed $Y \subset X$ of dimension $i+1$ such that the pushforward map $CH_i(Y) \rightarrow CH_i(X)$ is surjective. Then the pushforward map $CH^1(\tilde{Y}) \rightarrow CH_i(X)$ is surjective. But then, it is known that the Chow group of codimension-1 cycles on a smooth projective variety is representable.

Conversely, if $CH_i(X)$ is representable there exist a curve C and a correspondence $\Gamma \in CH_{i+1}(C \times X)$ such that $\Gamma_* : CH_0(C) \rightarrow CH_i(X)$ is surjective. Let Z be the support of a representative of Γ inside $C \times X$ and write $p : Z \rightarrow X$ for the projection. Then, if Y denotes the scheme-theoretic image of Z inside X , it is easy to check that the induced map $CH_i(Y) \rightarrow CH_i(X)$ is surjective. \square

Theorem 1.2. *Let X be a smooth projective variety of dimension n . If the Chow groups $CH_0(X_\Omega), \dots, CH_{\lfloor \frac{n-3}{2} \rfloor}(X_\Omega)$ have niveau ≤ 1 , then X satisfies Murre's conjectures. Moreover the motivic Lefschetz conjecture holds for X and hence Grothendieck's standard conjectures hold for X .*

We proved in [20] that if X is as in theorem 1.2 then X has a Chow-Künneth decomposition which is self-dual and which satisfies the motivic Lefschetz conjecture. Murre's conjectures for X are proved in [18]. Moreover, when X is odd-dimensional, the assumptions on X in theorem 1.2 can be weakened, see [18, §4.4.2]. If we strengthen the assumption on X , we have [20]

Theorem 1.3. *Let X be a smooth projective variety of dimension n . If the Chow groups $CH_0(X_\Omega), \dots, CH_{\lfloor \frac{n}{2} \rfloor - 1}(X_\Omega)$ have niveau ≤ 1 , then X is Kimura finite-dimensional.*

Laterveer [11] is mostly concerned with checking the validity of the Hodge conjecture for varieties with Chow groups having niveau ≤ 3 . He proves

Theorem 1.4 (Laterveer). *Let X be a smooth projective variety of dimension n . If the Chow groups $CH_0(X_\Omega), \dots, CH_{\lfloor \frac{n}{2} \rfloor - 2}(X_\Omega)$ have niveau ≤ 3 , then X satisfies the Hodge conjecture.*

The following statement deals with the Lefschetz standard conjecture.

Theorem 1.5. *Let X be a smooth projective variety of dimension n . If the Chow groups $CH_0(X_\Omega), \dots, CH_{\lfloor \frac{n-3}{2} \rfloor}(X_\Omega)$ have niveau ≤ 2 , then algebraic and homological equivalence agree on X and Grothendieck's Lefschetz standard conjecture holds for X .*

Proof. Since it is enough to prove the conclusion of the theorem for X_Ω , we may assume that X is defined over Ω . The statement about algebraic and homological equivalence agreeing was proved by Laterveer [11, 2.7]. Laterveer used the assumptions on the niveau of the Chow groups to show [11, 1.7] that the diagonal Δ_X admits a decomposition as follows : there exist closed and reduced subschemes $V_j, W^j \subset X$ with $\dim V_j \leq j + 2$ and $\dim W^j \leq n - j$, there exist correspondences $\Gamma_j \in CH_n(X \times X)$ for $0 \leq j \leq \lfloor \frac{n-3}{2} \rfloor$ and $\Gamma' \in CH_n(X \times X)$ such that each Γ_j is in the image of the pushforward map $CH_n(V_j \times W^j)$, Γ' is in the image of the pushforward map $CH_n(X \times W^{\lfloor \frac{n-1}{2} \rfloor})$, and

$$\Delta_X = \Gamma_1 + \dots + \Gamma_{\lfloor \frac{n-3}{2} \rfloor} + \Gamma'.$$

Given j such that $0 \leq j \leq \lfloor \frac{n-3}{2} \rfloor$, let \tilde{V}_j and \tilde{W}^j denote desingularisations of V_j and W^j respectively. The action of Γ_j on $H^k(X)$ then factors through $H^k(\tilde{V}_j)$ and through $H_{2n-k}(\tilde{W}^j)$. On the one hand, we have $H_{2n-k}(\tilde{W}^j) = H^{k-2j}(\tilde{W}^j)$ and hence if $k \leq 2j+1$ then the action of Γ_j on $H^k(X)$ factors through the H^0 or the H^1 of a smooth projective variety. Since the Lefschetz standard conjecture is true in degrees ≤ 1 , it follows that the action of Γ_j on $H^k(X)$ factors through the H_0 or the H_1 of a smooth projective variety. On the other hand, we have $H^k(\tilde{V}_j) = H_{4+2j-k}(\tilde{V}_j)$ and hence if $k \geq 2j+2$ then Γ_j factors through the H_0 , the H_1 or the H_2 of a smooth projective variety. Concerning the action of Γ' on $H^k(X)$, it factors through $H_{2n-k}(\tilde{W}^{\lfloor \frac{n-1}{2} \rfloor})$ which vanishes for dimension reasons if $k < n$ when n is odd and if $k < n-1$ when n is even. When n is even and $k = n-1$, the action of Γ' on $H^k(X)$ factors through the H_1 of a curve. Indeed this follows from a combination of the fact that it factors through $H_{n+1}(\tilde{W}^{\lfloor \frac{n-1}{2} \rfloor}) = H^1(\tilde{W}^{\frac{n-2}{2}})$ and of the validity of the Lefschetz standard conjecture in degree 1.

By the Lefschetz hyperplane theorem, we get that for $k < n$ the cohomology groups $H^k(X)$ are generated algebraically (that is through the action of correspondences) by the H_0 of points, the H_1 of curves and the H_2 of surfaces. The following proposition then makes it possible to conclude. \square

Proposition 1.6. *Let X be a smooth projective variety of dimension n and let $k > n$. Assume that $H_k(X) = \tilde{N}^{\lfloor k/2 \rfloor - 1} H_k(X)$. Here \tilde{N} denotes the niveau filtration of [18]. In other words assume that*

- *if k is odd, there exist a threefold Y_k and a correspondence $\Gamma_k \in CH_{\frac{k+3}{2}}(Y_k \times X)$ such that $(\Gamma_k)_* : H_3(Y_k) \rightarrow H_k(X)$ is surjective, and*
- *if k is even, there exist a surface Z_k and a correspondence $\Gamma_k \in CH_{\frac{k+2}{2}}(Z_k \times X)$ such that $(\Gamma_k)_* : H_2(Z_k) \rightarrow H_k(X)$ is surjective.*

Then X satisfies the Lefschetz standard conjecture in degree k .

Proof. See [19, Proposition 3.5]. \square

Remark 1.7. Note that theorem 1.5 does not seem to follow from the method of Arapura [1], since under our assumptions, it is not clear that the middle cohomology group of X is motivated by the cohomology of surfaces. Indeed, as can be read from the proof of the theorem, when n is even, the action of Γ' on $H^n(X)$ factors through $H_n(\widetilde{W}^{\lfloor \frac{n-1}{2} \rfloor}) = H^2(\widetilde{W}^{\frac{n-2}{2}})$. Although expected, it is not known if the H^2 of a smooth projective variety is motivated by the H^2 of a surface. Note that this would be the case were the Lefschetz standard conjecture known in degree 2.

2 Chow groups for varieties fibred by varieties with small Chow groups

Throughout this section we work over a field k of characteristic zero.

Proposition 2.1. *Let $f : X \rightarrow B$ be a projective dominant morphism onto an irreducible quasi-projective variety B of dimension d_B and let $H \hookrightarrow X$ be a linear section of dimension $\geq l + d_B$ such that f restricted to H is dominant. Assume that $CH_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all $0 < i \leq d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i . Here $X_{\eta_{D_i}}$ denotes the fibre of X over the generic point η_{D_i} of D_i . Then the natural map $\bigoplus_{b \in B} CH_l(X_b) \oplus CH_l(H) \rightarrow CH_l(X)$ is surjective.*

Proof. We prove the proposition by induction on d_B . If $d_B = 0$ then the statement is obvious. Let's thus consider a morphism $f : X \rightarrow B$ and a linear section $\iota : H \hookrightarrow X$ as in the statement of the proposition with $d_B > 0$. We have the localisation exact sequence

$$\bigoplus_{D \in B^1} CH_l(X_D) \rightarrow CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B}) \rightarrow 0.$$

Here, B^1 denotes the set of codimension-one closed irreducible subschemes of B . For any irreducible codimension-one subvariety $D \subset B$, the restriction of ι to $D \rightarrow B$ defines

a linear section $\iota_D : H_D \hookrightarrow X_D$ of dimension $\geq l + d_B - 1$ of X_D . It is easy to see that the restriction of $f : X \rightarrow B$ to $D \rightarrow B$ defines a dominant morphism $X_D \rightarrow D$ which together with the linear section ι_D satisfies the assumptions of the proposition. Therefore, by the induction hypothesis applied to $X_D \rightarrow D$, the map

$$\bigoplus_{d \in D} CH_l(X_d) \oplus CH_l(H_D) \rightarrow CH_l(X_D)$$

is surjective. This yields an exact sequence

$$\bigoplus_{b \in B} CH_l(X_b) \oplus \bigoplus_{D \in B^1} CH_l(H_D) \rightarrow CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B}) \rightarrow 0.$$

Since each of the proper inclusion maps $H_D \rightarrow X$ factors through $\iota : H \rightarrow X$, we see that the map $\bigoplus_{D \in B^1} CH_l(H_D) \xrightarrow{\oplus(\iota_D)^*} CH_l(X)$ factors through $\iota_* : CH_l(H) \rightarrow CH_l(X)$. In order to conclude, it is enough to prove that the composite map

$$CH_l(H) \rightarrow CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B})$$

is surjective. If $l < d_B$, then this is obvious. Let's then assume that $l \geq d_B$.

Let Y be an irreducible subvariety of H of dimension l such that the composite $Y \hookrightarrow X \rightarrow B$ is dominant. Because $CH_{l-d_B}(X_{\eta_B}) = \mathbf{Q}$ it is enough to show that the class of Y in $CH_l(X)$ maps to a non-zero element in $CH_{l-d_B}(X_{\eta_B})$. By the choice of Y , we see that the generic fibre Y_{η_B} of $f|_Y$ is an irreducible subvariety of X_{η_B} of dimension $l - d_B$ and hence its class in $CH_{l-d_B}(X_{\eta_B})$ is non-zero. The map $CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B})$ is obtained as the direct limit of the restriction maps $CH_l(X) \rightarrow CH_l(X_U)$ indexed by the open subsets $U \subset B$. By flat pullback, we see that the class of Y_{η_B} in $CH_{l-d_B}(X_{\eta_B})$ is the image of the class of Y by the map $CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B})$. \square

Here's an improvement:

Proposition 2.2. *Let $f : X \rightarrow B$ be a projective dominant morphism onto an irreducible quasi-projective variety B of dimension d_B and let $H \hookrightarrow X$ be a linear section of dimension $\geq l + d_B$ such that f restricted to H is dominant. Assume that:*

- $CH_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all i such that $0 < i < d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i . Here $X_{\eta_{D_i}}$ denotes the fibre of X over the generic point η_{D_i} of D_i .
- $CH_{l-d_B}(X_{\eta_B})$ is finitely generated. Here X_{η_B} is the generic fibre of $X \rightarrow B$.

Then there exist finitely many closed subschemes \mathcal{Z}_j of X of dimension l such that the natural map $\bigoplus_j CH_l(\mathcal{Z}_j) \oplus \bigoplus_{b \in B} CH_l(X_b) \oplus CH_l(H) \rightarrow CH_l(X)$ is surjective.

Proof. If $d_B = 0$ then the statement is obvious. Let's thus consider a morphism $f : X \rightarrow B$ and a linear section $\iota : H \hookrightarrow X$ as in the statement of the proposition with $d_B > 0$. As in the proof of proposition 2.1 we have the localisation exact sequence

$$\bigoplus_{D \in B^1} CH_l(X_D) \rightarrow CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B}) \rightarrow 0.$$

Each of the morphisms $X_D \rightarrow D$ satisfies the assumptions of proposition 2.1 and by the same arguments as in the proof of proposition 2.1 we get that the image of the map $\bigoplus_{b \in B} CH_l(X_b) \oplus CH_l(H) \rightarrow CH_l(X)$ contains the image of the map $\bigoplus_{D \in B^1} CH_l(X_D) \rightarrow CH_l(X)$. Let now Z_j be finitely many closed subschemes of X_{η_B} whose classes $[Z_j] \in CH_{l-d_B}(X_{\eta_B})$ generate $CH_{l-d_B}(X_{\eta_B})$. By surjectivity of the map $CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B})$ there are cycles $\alpha_j \in CH_l(X)$ that map to $[Z_j]$. If \mathcal{Z}_j is the support in X of any representative of α_j , we then have a surjective map $\bigoplus_j CH_l(\mathcal{Z}_j) \rightarrow CH_l(X) \rightarrow CH_{l-d_B}(X_{\eta_B})$. It is then clear that the map $\bigoplus_j CH_l(\mathcal{Z}_j) \oplus \bigoplus_{b \in B} CH_l(X_b) \oplus CH_l(H) \rightarrow CH_l(X)$ is surjective. \square

2.1 Varieties fibred by varieties with Chow groups generated by a linear section

Proposition 2.3. *Let $f : X \rightarrow B$ be a dominant projective morphism onto a quasi-projective variety B of dimension d_B . Assume that $CH_l(X_b) = \mathbf{Q}$ for all closed points $b \in B$. Then, if $H \hookrightarrow X$ is a linear section of dimension $\geq l + d_B$ such that f restricted to H is dominant, we have*

$$\mathrm{Im} \left(\bigoplus_{b \in B} CH_l(X_b) \rightarrow CH_l(X) \right) \subseteq \mathrm{Im} (CH_l(H) \rightarrow CH_l(X)).$$

Proof. Let b be a closed point of B and fix $H \hookrightarrow X$ a linear section of dimension $\geq l + d_B$ such that f restricted to H is dominant. Let Z_l be an irreducible closed subscheme of X of dimension l which is supported on X_b . Since $f|_H : H \rightarrow B$ is a dominant projective morphism, its fibre H_b over b is non-empty and has dimension $\geq l$. By assumption $CH_l(X_b) = \mathbf{Q}$, so that a rational multiple of $[Z_l]$ is rationally equivalent to an irreducible closed subscheme of H_b of dimension l . Therefore $[Z_l] \in CH_l(X_b)$ belongs to the image of the natural map $CH_l(H_b) \rightarrow CH_l(X_b)$. It is then not hard to conclude. \square

Remark 2.4. It is interesting to decide whether or not it is possible to parametrise such l -cycles by a variety of dimension d_B . See theorem 2.8.

Theorem 2.5. *Let $f : X \rightarrow B$ be a dominant projective morphism onto a quasi-projective variety B of dimension d_B . Assume that $CH_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all $0 \leq i \leq d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i , where η_{D_i} is the generic point of D_i . Then, if $H \hookrightarrow X$ is a linear section of dimension $\geq l + d_B$ such that f restricted to H is dominant, we have*

$$CH_l(X) = \mathrm{Im} (CH_l(H) \rightarrow CH_l(X)).$$

In particular, $CH_l(X)$ has niveau $\leq d_B$.

Proof. This is a combination of proposition 2.1 and proposition 2.3. \square

2.2 An argument involving relative Hilbert schemes

Let $f : X \rightarrow B$ be a generically smooth projective dominant morphism defined over a field $k \subseteq \mathbf{C}$ onto a smooth quasi-projective variety B . Let $B^\circ \subseteq B$ be the smooth locus of f and let $f^\circ : X^\circ \rightarrow B^\circ$ be the pullback of $f : X \rightarrow B$ along the open inclusion $B^\circ \hookrightarrow B$ so that we have a cartesian square

$$\begin{array}{ccc} X^\circ & \xrightarrow{\quad} & X \\ f^\circ \downarrow & & \downarrow f \\ B^\circ & \xrightarrow{\quad} & B. \end{array}$$

We assume that there is a non-negative integer l such that for all closed points $b \in B^\circ$ the cycle class map $CH_l(X_b) \rightarrow H_{2l}(X_b)$ is an isomorphism.

For K a field containing the base field k , we define for $d \geq 0$

$$E_d(K) := \{b \in B^\circ(K) : CH_l(X_b) \text{ is generated by closed subschemes of } X_b \text{ of degree } \leq d\}.$$

By assumption on the cycle class map, $CH_l(X_b)$ is finitely generated for all $b \in B^\circ(K)$. Hence the set $E_d(K)$ is non-empty for d large. Clearly,

$$B^\circ(K) = \bigcup_{d \geq 0} E_d(K).$$

Proposition 2.6. *E_d has naturally the structure of a subscheme of B° . Moreover E_d is locally closed inside B° .*

Proof. Let $\pi : \text{Hilb}_l^{\leq d}(X/B) \rightarrow B$ be the relative Hilbert scheme whose fibres over the points b in B parametrise the closed subschemes of X_b of dimension l and degree $\leq d$. Let's then write

$$\text{Hilb}_l^{\leq d}(X/B) := \bigcup_i \mathcal{H}_i$$

where the \mathcal{H}_i 's are the irreducible components of $\text{Hilb}_l^{\leq d}(X/B)$. Let $p : \mathcal{C}_l^{\leq d} \rightarrow \text{Hilb}_l^{\leq d}(X/B)$ be the universal family over $\text{Hilb}_l^{\leq d}(X/B)$. We have the following commutative diagram, where all morphisms involved are proper:

$$\begin{array}{ccc} \mathcal{C}_l^{\leq d} & \xrightarrow{q} & X \\ p \downarrow & & \swarrow \\ \text{Hilb}_l^{\leq d}(X/B) & & \\ \pi \downarrow & \swarrow & \\ B & & \end{array}$$

If \mathcal{H}_i is one of the irreducible components of $\text{Hilb}_l^{\leq d}(X/B)$ and if s and s' are two K -points of \mathcal{H}_i then the classes of the fibres $p^{-1}(s)$ and $p^{-1}(s')$ define, up to a rational scalar, algebraically equivalent cycles in $CH_l(\mathcal{C}_l^{\leq d})$, so that there exists $a \in \mathbf{Q}$ such that

$$cl(q_*[p^{-1}(s)]) = a \cdot cl(q_*[p^{-1}(s')]) \in H_{2l}(X).$$

For $t \in E_d(K)$, we define

$$A_{d,t} := \{\mathcal{H}_i : t \in \pi(\mathcal{H}_i)\}.$$

Let \mathcal{E} be a subset of $A_{d,t}$. We say that \mathcal{E} spans $H_{2l}(X_t)$ if $H_{2l}(X_t)$ is spanned by the (finite) set $\{cl([p^{-1}(s)]) : s \in \mathcal{H}_i, \mathcal{H}_i \in \mathcal{E}, \pi(s) = t\}$. We define

$$\mathcal{A}_{d,t} := \{\mathcal{E} \in \mathcal{P}(A_{d,t}) : \mathcal{E} \text{ spans } H_{2l}(X_t)\}.$$

This set is non-empty because we are assuming that the cycle map $CH_l(X_t) \rightarrow H_{2l}(X_t)$ is an isomorphism.

We then define

$$Z_{d,t} := \bigcup_{\mathcal{E} \in \mathcal{A}_{d,t}} \bigcap_{\mathcal{H} \in \mathcal{E}} \pi(\mathcal{H}).$$

The morphism π is proper so that $Z_{d,t}$ is Zariski-closed inside B . Therefore, since K was an arbitrary extension of k , we will be done if we prove that for any $t \in E_d(K)$, there exists an open neighbourhood U_t of t inside B° such that $U_t(K) \cap E_d(K) = U_t(K) \cap Z_{d,t}(K)$. Let's first notice that because $R_l f_* \mathbf{Q}$ is a local system on B° which is spanned at t by any choice of \mathcal{E} in $\mathcal{A}_{d,t}$, there is a Zariski open neighborhood V_t of t inside B° such that, for $t' \in Z_{d,t}(K) \cap V_t(K)$, $H_{2l}(X_{t'})$ is spanned at t' by any choice of \mathcal{E} in $\mathcal{A}_{d,t}$, i.e. such that we have the inclusion $Z_{d,t}(K) \cap V_t(K) \subseteq E_d(K)$. Let's then define

$$V'_t := B^\circ - \bigcup_{\mathcal{H}_i \notin A_{d,t}} \pi(\mathcal{H}_i) \cap B^\circ.$$

It is then not too hard to check that $E_d(K) \cap V'_t(K) \subseteq Z_{d,t}(K)$. Thus, if we define $U_t := V_t \cap V'_t$, then $U_t(K) \cap E_d(K) = U_t(K) \cap Z_{d,t}(K)$ and E_d has naturally the structure of a subscheme of B° and considered as such it is locally closed inside B° . \square

Corollary 2.7. $B^\circ = E_d$ for d large enough.

Proof. By proposition 2.6, B° is the countable union of the subschemes E_d . For all $d < d'$, E_d is clearly a subscheme of $E_{d'}$. Therefore B° must be equal to E_d for some d . \square

Theorem 2.8. Let $f : X \rightarrow B$ be a generically smooth projective dominant morphism onto a smooth quasi-projective variety B of dimension d_B . Let $B^\circ \subseteq B$ be the smooth locus of f . Assume that there is an integer $l \leq d_X - d_B$ such that for all closed points $b \in B^\circ$ the cycle class map $CH_l(X_b) \rightarrow H_{2l}(X_b)$ is an isomorphism. Then $\text{Im}(\bigoplus_{b \in B^\circ} CH_l(X_b) \rightarrow CH_l(X))$ is supported on a closed subvariety of X of dimension $d_B + l$.

If moreover X is smooth, then there exist a smooth quasi-projective variety \tilde{B} of dimension d_B and a correspondence $\Gamma \in CH_{d_B+l}(\tilde{B} \times X)$ such that $\Gamma_* : CH_0(\tilde{B}) \rightarrow CH_l(X)$ is well-defined and

$$\mathrm{Im} \left(\bigoplus_{b \in B^\circ} CH_l(X_b) \rightarrow CH_l(X) \right) \subseteq \mathrm{Im} (\Gamma_* : CH_0(\tilde{B}) \rightarrow CH_l(X)).$$

Proof. Let d be an integer such that $B^\circ = E_d$ and let \mathcal{H}_i be the irreducible components of $Hilb_l^{\leq d}(X/B)$. Let then $\tilde{\mathcal{H}}_i \rightarrow \mathcal{H}_i$ be resolutions of the \mathcal{H}_i 's and for all i pick a smooth linear section $\tilde{B}_i \rightarrow \tilde{\mathcal{H}}_i$ of dimension d_B that dominates B . Consider then $p_i : (\mathcal{C}_l^{\leq d})|_{\tilde{B}_i} \rightarrow \tilde{B}_i$ the pullback of the universal family $p : \mathcal{C}_l^{\leq d} \rightarrow Hilb_l^{\leq d}(X/B)$ along $\tilde{B}_i \hookrightarrow \tilde{\mathcal{H}}_i \rightarrow \mathcal{H}_i \hookrightarrow Hilb_l^{\leq d}(X/B)$. For each i , we have the following picture

$$\begin{array}{ccc} (\mathcal{C}_l^{\leq d})|_{\tilde{B}_i} & \xrightarrow{q_i} & X \\ p_i \downarrow & & \\ \tilde{B}_i & & \end{array}$$

and it is clear that

$$\mathrm{Im} \left(\bigoplus_{b \in B^\circ} CH_l(X_b) \rightarrow CH_l(X) \right) \subseteq \sum_i \mathrm{Im} ((q_i)_* : CH_l((\mathcal{C}_l^{\leq d})|_{\tilde{B}_i}) \rightarrow CH_l(X))$$

so that the group on the left-hand side is supported on the union of the scheme-theoretic images of the morphisms q_i .

If X is smooth we define $\Gamma_i \in CH_{d_B+l}(\tilde{B}_i \times X)$ to be the class of the image of $(\mathcal{C}_l^{\leq d})|_{\tilde{B}_i}$ inside $\tilde{B}_i \times X$. Because $q : \mathcal{C}_l^{\leq d} \rightarrow X$ is proper, Γ_i has a representative which is proper over X . It is therefore possible [4, Rk 16.1] to define maps $(\Gamma_i)_* : CH_0(\tilde{B}_i) \rightarrow CH_l(X)$ for all i . Finally we define \tilde{B} to be the disjoint union of the \tilde{B}_i 's and $\Gamma \in CH_{d_B+l}(\tilde{B} \times X)$ to be the class of the disjoint union of the Γ_i 's. \square

2.3 Varieties fibred by varieties with small Chow groups

Theorem 2.9. *Let $f : X \rightarrow C$ be a generically smooth projective dominant map onto a smooth curve. Assume that*

- $CH_l(X_c)$ is finitely generated for all closed points $c \in C$,
- $CH_l(X_c) \rightarrow H_{2l}(X_c)$ is an isomorphism for a general closed point $c \in C$,
- $CH_{l-1}(X_\eta)$ is finitely generated, where η is the generic point of C .

Then $CH_l(X)$ has niveau ≤ 1 .

Proof. We have the localisation exact sequence

$$\bigoplus_{c \in C} CH_l(X_c) \longrightarrow CH_l(X) \longrightarrow CH_{l-1}(X_\eta) \longrightarrow 0.$$

Let Z_1, \dots, Z_n be irreducible closed subschemes of X_η of dimension $l-1$ that span $CH_{l-1}(X_\eta)$ and let $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ be closed subschemes of X of dimension l that restrict to Z_1, \dots, Z_n in X_η . Then by flat pullback the class of \mathcal{Z}_j in $CH_l(X)$ maps to the class of Z_j in $CH_{l-1}(X_\eta)$ so that the composite map $\bigoplus_{j=1}^n CH_l(\mathcal{Z}_j) \rightarrow CH_l(X) \rightarrow CH_{l-1}(X_\eta)$ is surjective.

Let $U \subseteq C$ be a Zariski-open subset of C such that for all closed points $c \in U$ the cycle class map $CH_l(X_c) \rightarrow H_{2l}(X_c)$ is an isomorphism. Up to shrinking U , we may assume that $f|_U : X|_U \rightarrow U$ is smooth. We may then apply theorem 2.8 to get a closed subscheme $\iota : D \hookrightarrow X$ of dimension $l+1$ such that $\iota_* CH_l(D) \supseteq \text{Im} \left(\bigoplus_{c \in U} CH_l(X_c) \rightarrow CH_l(X) \right)$.

As such, we have a surjective map

$$\bigoplus_{j=1}^n CH_l(\mathcal{Z}_j) \oplus \bigoplus_{c \in C-U} CH_l(X_c) \oplus CH_l(D) \longrightarrow CH_l(X)$$

and it is straightforward to conclude. \square

Remark 2.10. In the above theorem, we could change the assumption “ $CH_{l-1}(X_\eta)$ is finitely generated” by “ $CH_{l-1}(X_\eta)$ has niveau ≤ 1 ” to get that $CH_l(X)$ has niveau ≤ 1 . However, this is irrelevant because of the Hard Lefschetz theorem, in view of the Bloch-Beilinson conjectures.

The next theorem is a generalisation of theorem 2.9 to the case when the base variety B has dimension greater than 1.

Theorem 2.11. *Let $f : X \rightarrow B$ be a generically smooth projective dominant map onto a smooth quasi-projective variety B of dimension d_B . Assume that the singular locus of f in B is finite and let U be the maximal Zariski-open subset of B over which f is smooth. Assume also that*

- $CH_l(X_b)$ is finitely generated for all closed points $b \in B$,
- $CH_l(X_b) \rightarrow H_{2l}(X_b)$ is an isomorphism for all closed points $b \in U$,
- $CH_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all i such that $0 < i < d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i , where η_{D_i} is the generic point of D_i .
- $CH_{l-d_B}(X_{\eta_B})$ is finitely generated. Here X_{η_B} is the generic fibre of f .

Then $CH_l(X)$ has niveau $\leq d_B$.

Proof. Let $H \hookrightarrow X$ be a linear section of dimension $\geq l + d_B$ such that f restricted to H is dominant. Thanks to proposition 2.2, there are finitely many closed subschemes \mathcal{Z}_j of X of dimension l such that the natural map $\bigoplus_j CH_l(\mathcal{Z}_j) \oplus \bigoplus_{b \in B} CH_l(X_b) \oplus CH_l(H) \rightarrow CH_l(X)$ is surjective. By theorem 2.8, there exists a closed subscheme $\iota : \tilde{B} \hookrightarrow X$ of dimension $d_B + l$ such that the image of the map $\bigoplus_{b \in U} CH_l(X_b) \rightarrow CH_l(X)$ is contained in the image of the map $\iota_* : CH_l(\tilde{B}) \rightarrow CH_l(X)$. Therefore the map

$$\bigoplus_j CH_l(\mathcal{Z}_j) \oplus CH_l(\tilde{B}) \oplus \bigoplus_{b \in B-U} CH_l(X_b) \oplus CH_l(H) \longrightarrow CH_l(X)$$

is surjective. It is then straightforward to conclude. \square

In the case of zero-cycles, when X is smooth, there is a less restrictive statement that only involves the fibres of f over some Zariski open subset of the base. However, the assumptions of theorem 2.9 or 2.12 made on all closed fibres of f cannot be relaxed to a general fibre of f . Consider for example the blow-up of a projective bundle over a curve C along a subvariety Y contained in one of the closed fibres to get a morphism $X \rightarrow C$ such that $CH_1(X_c) = \mathbf{Q}$ for all but one closed points $c \in C$, $CH_0(X_\eta) = \mathbf{Q}$ and $CH_1(X) \simeq CH_0(Y) \oplus CH_0(C) \oplus CH_1(C)$. The following is proved in [17, Theorem 1.3] when B is projective.

Theorem 2.12. *Let $f : X \rightarrow B$ be a projective dominant morphism from a smooth variety X onto a smooth quasi-projective variety B . Assume that $CH_0(X_b) = \mathbf{Q}$ for a general point $b \in B$. Then $f_* : CH_0(X) \rightarrow CH_0(B)$ is an isomorphism and there exists a correspondence $\Gamma \in CH_{d_B}(B \times X)$ such that $\Gamma_* : CH_0(B) \rightarrow CH_0(X)$ is well-defined and is the inverse of f_* .*

Proof. By Bertini, let $\sigma : H \rightarrow X$ be a smooth linear section of X of dimension d_B such that $\pi := f \circ \sigma$ is dominant. In particular, π is proper and generically finite, say of degree n . The projection formula then shows that, for $\alpha \in CH_0(B)$, $\pi_* \circ \pi^* \alpha = n\alpha$. Therefore we may conclude if we can show that $\sigma_* \circ \pi^* : CH_0(B) \rightarrow CH_0(X)$ is surjective, in which case $\Gamma := \Gamma_\sigma \circ {}^t\Gamma_\pi$, which is well-defined, will do. Here Γ_π denotes the class of the graph of π in $CH_{d_B}(H \times B)$ and ${}^t\Gamma_\pi$ its transpose, and the composite $\Gamma := \Gamma_\sigma \circ {}^t\Gamma_\pi$ is well-defined by Fulton's refined intersection product [4, §8.1 & Remark 16.1]. Let U be a dense open subset of B over which π is finite and such that for all closed point u in U , $CH_0(X_u) = \mathbf{Q}$. Let p be a closed point in X . By Chow's moving lemma, the zero-cycle $[p]$ is rationally equivalent to a zero-cycle $\sum a_i [p_i]$ where each closed point p_i belongs to the open subset $X|_U$ of X . It is then easy to see that $\sigma_* \pi^* f_* [p_i]$ has non-zero degree and is rationally equivalent to $[p_i]$ up to a rational scalar. Hence $\sigma_* \pi^* f_* : CH_0(X) \rightarrow CH_0(X)$ is surjective. \square

3 The complex case

Ultimately our aim is to give examples of varieties for which the results of the previous section apply. For this purpose we need to relate geometric properties of fibres over any points to geometric properties of fibres over closed points. The following statement was communicated to me by Burt Totaro.

Lemma 3.1. *Let $f : X \rightarrow B$ be a morphism of complex varieties with B irreducible and let F be a geometric generic fibre of f . Then there is a subset $U \subseteq B(\mathbf{C})$ which is a countable intersection of nonempty Zariski open subsets such that for each point $p \in U$, there is an isomorphism from the field \mathbf{C} to the field $\overline{\mathbf{C}(B)}$ such that this isomorphism turns the scheme X_p over \mathbf{C} into the scheme F over $\overline{\mathbf{C}(B)}$.*

Proof. There exist a countable subfield $K \subset \mathbf{C}$ and varieties X_0 and B_0 defined over K together with a K -morphism $f_0 : X_0 \rightarrow B_0$ such that $f = f_0 \times_K \mathbf{C}$. Let $p : \text{Spec } \mathbf{C} \rightarrow B$ be a complex point of B such that the composite map $p : \text{Spec } \mathbf{C} \rightarrow B \rightarrow B_0$ is dominant,

i.e. such that p factors through the generic point $\eta_{B_0} = \text{Spec } K(B_0) \rightarrow B_0$ of B_0 . Now there is an isomorphism $\overline{\mathbf{C}(B)} \xrightarrow{\simeq} \mathbf{C}$ of fields over $K(B_0)$. In particular there is a commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathbf{C} & \xrightarrow{p} & B \\ \downarrow \simeq & \nearrow & \\ \overline{\eta_B} & & \end{array}$$

where $\overline{\eta_B} \rightarrow B$ denotes a geometric generic point of B such that $X_{\overline{\eta_B}} = F$. We thus see that X_p identifies with F after pull-back by the isomorphism $\text{Spec } \mathbf{C} \rightarrow \overline{\eta_B}$.

Let's then define U as being the complex points of $B = B_0 \times_K \mathbf{C}$ that do not lie above a proper Zariski-closed subset of B_0 , i.e. U is the set of points $\text{Spec } \mathbf{C} \rightarrow B$ such that the composite map $\text{Spec } \mathbf{C} \rightarrow B \rightarrow B_0$ factors through the generic point $\eta_{B_0} \rightarrow B_0$ of B_0 . \square

Proposition 3.2. *Let $f : X \rightarrow B$ be a morphism of complex varieties with B irreducible and let F be the geometric generic fibre of f . Then there is a subset $U \subseteq B(\mathbf{C})$ which is a countable intersection of nonempty Zariski open subsets such that for each point $p \in U$, $CH_i(X_p)$ is isomorphic to $CH_i(F)$ for all i .*

Proof. Thanks to lemma 3.1, there is a U as in the proposition such that all the fibres X_p for $p \in U$ are isomorphic as abstract schemes to the geometric generic fibre F . The proposition then follows from the fact that, by definition, the Chow groups of a variety X over a field only depend on X as a scheme. \square

Here's a lemma concerned with checking the second point in the assumptions of theorems 2.9 and 2.11 in concrete situations.

Lemma 3.3. *Let X be a smooth projective complex variety. Assume that $CH_i(X)$ is finitely generated for all $i \leq l$. Then the cycle class map $CH_l(X) \rightarrow H_{2l}(X)$ is an isomorphism.*

Proof. The proof follows the same pattern as the proof of [21, Theorem 3.4] once it is noticed that if $CH_i(X)$ is finitely generated then it is representable. Concretely, by following the method of proof of [21, Theorem 3.4], we get that the Chow motive of X is isomorphic to $\mathbb{1} \oplus \mathbb{1}(1)^{b_2} \oplus \dots \oplus \mathbb{1}(l)^{b_{2l}} \oplus N(l+1)$ where b_i is the i -th Betti number of X and N is an effective motive. From here, it is straightforward that the cycle class map $CH_i(X) \rightarrow H_{2i}(X)$ is an isomorphism for all $i \leq l$. \square

In this context, as a corollary of theorem 2.5, we obtain

Theorem 3.4. *Let $f : X \rightarrow B$ be a complex projective dominant morphism onto a complex quasi-projective variety B of dimension d_B . Assume that $CH_i(X_b) = \mathbf{Q}$ for all $i \leq l$ and all closed point $b \in B$. Then $CH_i(X)$ has niveau $\leq d_B$ for all $i \leq l$.*

Proof. Let D be an irreducible subvariety of X and let $\bar{\eta}_D \rightarrow D$ be a geometric generic point of D . Let $i \leq l$. By proposition 3.2, there is a closed point $d \in D$ such that $CH_i(X_{\bar{\eta}_D})$ is isomorphic to $CH_i(X_d)$. By assumption $CH_i(X_d) = \mathbf{Q}$. Therefore $CH_i(X_{\bar{\eta}_D}) = \mathbf{Q}$, too. The pullback map $CH_i(X_{\eta_D}) \rightarrow CH_i(X_{\bar{\eta}_D})$ is injective. Hence $CH_i(X_{\eta_D}) = \mathbf{Q}$. Now we can apply theorem 2.5 to get that $CH_i(X)$ has niveau $\leq d_B$. \square

As a corollary of theorem 2.9, we obtain

Theorem 3.5. *Let $f : X \rightarrow C$ be a complex generically smooth projective dominant morphism onto a smooth complex curve. Assume that $CH_i(X_c)$ is finitely generated for all closed points $c \in C$ and all $i \leq l$, then $CH_i(X)$ has niveau ≤ 1 for all $i \leq l$.*

Proof. Let D be an irreducible component of C . The same arguments as in the proof of theorem 3.4 show that $CH_i(X_{\eta_D})$ is finitely generated for all $i \leq l$. Let then $U \subseteq C$ be the smooth locus of f . It is an open dense subset of C . Then, for $c \in U$, the closed fibre X_c is smooth and the groups $CH_i(X_c)$ are finitely generated for all $i \leq l$. By lemma 3.3 the cycle class maps $CH_i(X_c) \rightarrow H_{2i}(X_c)$ are isomorphisms for all $c \in U$ and all $i \leq l$. Now we can apply theorem 2.9 to get that $CH_i(X)$ has niveau ≤ 1 . \square

And, as a corollary of theorem 2.11, we obtain

Theorem 3.6. *Let $f : X \rightarrow B$ be a complex generically smooth projective dominant morphism onto a smooth quasi-projective complex variety B of dimension d_B . Assume that the singular locus of f in B is finite. Assume also that*

- $CH_i(X_b)$ is finitely generated for all closed points $b \in B$ and all $i \leq l$,
- $CH_i(X_b) = \mathbf{Q}$ for all but finitely many closed points $b \in B$ and all $i < l$.

Then $CH_i(X)$ has niveau $\leq d_B$ for all $i \leq l$.

Proof. Let D be an irreducible subvariety of X of positive dimension and let $\bar{\eta}_D \rightarrow D$ be a geometric generic point of D . Let $i < l$. By proposition 3.2, for a very general closed point $d \in D$, the group $CH_i(X_{\bar{\eta}_D})$ is isomorphic to $CH_i(X_d)$. The second assumption of the theorem implies that $CH_i(X_d) = \mathbf{Q}$ for a general closed point of $d \in D$. As in the proof of theorem 3.5, we get $CH_i(X_{\eta_D}) = \mathbf{Q}$ for all $i < l$. Then, if we denote by U the smooth locus of f , by lemma 3.3, the cycle class maps $CH_i(X_b) \rightarrow H_{2i}(X_b)$ are isomorphisms for all $b \in U$ and all $i \leq l$. Now we can apply theorem 2.11 to get that $CH_i(X)$ has niveau $\leq d_B$ for all $i \leq l$. \square

4 Applications

4.1 Varieties fibred by complete intersections

4.1.1 Chow groups of complete intersections

Let k be a field. As explained by Esnault-Levine-Viehweg in the introduction of [3], it is expected from general conjectures on algebraic cycles, that if $X \subset \mathbf{P}_k^n$ is a complete

intersection of multidegree $d_1 \geq \dots \geq d_r \geq 2$, then $CH_l(X) = \mathbf{Q}$ for all $l < \lfloor \frac{n - \sum_{i=2}^r d_i}{d_1} \rfloor$, see also Paranjape [14] and Schoen [15]. If there is no proof of the above for the moment, the following theorem however was proved by Esnault-Levine-Viehweg [3].

Theorem 4.1 (Esnault-Levine-Viehweg). *Let $X \subset \mathbf{P}_k^n$ be a complete intersection of multidegree $d_1 \geq \dots \geq d_r \geq 2$. If either $d_1 \geq 3$ or $r \geq l + 1$, assume that*

$$\sum_{i=1}^r \binom{l + d_i}{l + 1} \leq n.$$

If $d_1 = \dots = d_r = 2$ and $r \leq l$, assume that

$$\sum_{i=1}^r \binom{l + d_i}{l + 1} = r(l + 2) \leq n - l + r - 1.$$

Then $CH_{l'}(X) = \mathbf{Q}$ for all $0 \leq l' \leq l$. □

4.1.2 Varieties fibred by very low degree complete intersections

Varieties fibred by quadric hypersurfaces. Let $Q \subset \mathbf{P}^n$ be a quadric hypersurface. Then $CH_l(Q) = \mathbf{Q}$ for all $l < \frac{\dim Q}{2}$ and if $\dim X$ is even, $CH_{\frac{\dim Q}{2}}(Q)$ is finitely generated.

Theorem 4.2. *Let $f : X \rightarrow B$ be a dominant morphism between smooth projective complex varieties whose fibres are quadric hypersurfaces.*

- *If $\dim B \leq 1$, then X is finite-dimensional in the sense of Kimura and satisfies Murre's conjectures as well as the motivic Lefschetz conjecture.*
- *If $\dim B \leq 2$, then X satisfies Grothendieck's standard conjectures.*
- *If $\dim B \leq 3$, then X satisfies the Hodge conjecture.*

Proof. By theorem 4.1, X satisfies the assumptions of theorem 3.4 with $l = \lfloor \frac{\dim Q - 1}{2} \rfloor$. Thus the Chow groups $CH_0(X), CH_1(X), \dots, CH_{\lfloor \frac{\dim Q - 1}{2} \rfloor}(X)$ have niveau $\leq d_B$. We can therefore conclude by theorems 1.2, 1.3, 1.4 and 1.5. □

Remark 4.3. In theorem 4.2, if $\dim B \leq 2$ and if $f : X \rightarrow B$ is moreover assumed to be flat, then it is proved in [16] that X satisfies Murre's conjectures as well as the motivic Lefschetz conjecture.

Varieties fibred by cubic hypersurfaces. Let $X \subset \mathbf{P}^n$ be a cubic hypersurface. Then

- $CH_0(X) = \mathbf{Q}$ for $\dim X \geq 2$.
- $CH_1(X) = \mathbf{Q}$ for $\dim X \geq 5$.
- $CH_2(X) = \mathbf{Q}$ for $\dim X \geq 8$.

Notice that theorem 4.1 only gives $CH_2(X) = \mathbf{Q}$ for $\dim X \geq 9$. The bound on the dimension of X was improved to $\dim X = 8$ by Otwinowska [13].

Theorem 4.4. *Let $X \rightarrow B$ be a dominant morphism between smooth projective complex varieties whose fibres are cubic hypersurfaces.*

- *If $\dim X = 6$ and $\dim B = 1$, then X satisfies Murre's conjectures and the motivic Lefschetz conjecture (and hence Grothendieck's standard conjectures).*
- *If $\dim X = 7$ and if $\dim B \leq 2$, then X satisfies the Hodge conjecture.*
- *If $\dim X = 9$ and if $\dim B \leq 1$, then X satisfies the Hodge conjecture.*

Proof. We use theorem 3.4 as in the proof of theorem 4.2. In the first case, we get that $CH_0(X)$ and $CH_1(X)$ have niveau ≤ 1 . Therefore we can conclude by theorem 1.2. In the second case we get that $CH_0(X)$ and $CH_1(X)$ have niveau ≤ 2 and in the third case we get that $CH_0(X)$, $CH_1(X)$ and $CH_2(X)$ have niveau ≤ 1 . Therefore we can conclude in both cases by theorem 1.4. \square

Varieties fibred by complete intersections of bidegree $(2, 2)$. Let $X \subset \mathbf{P}^n$ be the complete intersection of two quadrics. By theorem 4.1, $CH_0(X) = \mathbf{Q}$; and if $\dim X \geq 4$, then $CH_1(X) = \mathbf{Q}$.

Theorem 4.5. *Let $f : X \rightarrow B$ be a dominant morphism between smooth projective complex varieties whose fibres are complete intersections of bidegree $(2, 2)$.*

- *If $\dim B \leq 1$ and $\dim X \leq 5$, then X is finite-dimensional in the sense of Kimura.*
- *If $\dim B \leq 1$ and $\dim X \leq 6$, then X satisfies Murre's conjectures.*
- *If $\dim B \leq 2$ and $\dim X \leq 6$, then X satisfies Grothendieck's standard conjectures.*
- *If $\dim B \leq 3$ and $\dim X \leq 7$, then X satisfies the Hodge conjecture.*

Proof. The variety X satisfies the assumptions of theorem 3.4 with $l = 1$ for $\dim X - \dim B \geq 4$ and with $l = 0$ in any case. Thus the Chow group $CH_0(X)$ has niveau $\leq d_B$ and $CH_1(X)$ has niveau $\leq d_B$ for $\dim X - \dim B \geq 4$. We can therefore conclude by theorems 1.2, 1.3, 1.4 and 1.5. \square

Remark 4.6. In theorem 4.5, if $\dim B = 2$, $\dim X = 6$, and if $f : X \rightarrow B$ is moreover assumed to be flat, then it is proved in [16] that X satisfies Murre's conjectures.

Varieties fibred by complete intersections of bidegree $(2, 3)$. Let $X \subset \mathbf{P}^n$ be the complete intersection of a quadric and of a cubic. If $\dim X \geq 6$, then Hirschowitz and Iyer [6] showed $CH_l(X) = \mathbf{Q}$ for $l \leq 1$. (The result of Esnault-Levine-Viehweg only says that $CH_l(X) = \mathbf{Q}$ for $l \leq 1$ when $\dim X \geq 7$).

Theorem 4.7. *Let $X \rightarrow C$ be a dominant morphism from a smooth projective complex variety X to a smooth projective complex curve C whose fibres are complete intersections of bidegree $(2, 3)$ of dimension 6. Then X satisfies the Hodge conjecture.*

Proof. By theorem 3.4, we see that the Chow groups $CH_0(X)$ and $CH_1(X)$ have niveau ≤ 1 . We can thus conclude by theorem 1.4. \square

4.2 Varieties fibred by cellular varieties

Let $f : X \rightarrow B$ be a complex dominant morphism from a smooth projective variety X to a smooth projective variety B such that for all closed points $p : \operatorname{Spec} \mathbf{C} \rightarrow B$, X_p is a cellular variety (not necessarily smooth). In other words, X is a smooth projective complex variety fibred by cellular varieties over B . For example, X could be a rational homogeneous bundle over B , i.e. f is smooth and each fibre of f is isomorphic to a rational homogeneous variety.

Theorem 4.8. *Let $f : X \rightarrow B$ be a dominant morphism between smooth projective complex varieties such that for all closed points $p : \operatorname{Spec} \mathbf{C} \rightarrow B$, X_p is a cellular variety.*

- *Assume B is a curve, then X is Kimura finite-dimensional and X satisfies the motivic Lefschetz conjecture and Murre's conjectures.*
- *Assume $\dim B \leq 2$ and $\dim X \leq 6$. If f is connected and smooth away from finitely many points in B , then X satisfies the Lefschetz standard conjecture and hence the standard conjectures.*
- *Assume $\dim B \leq 3$ and $\dim X \leq 7$. If f is connected and smooth away from finitely many points in B , then X satisfies the Hodge conjecture.*

Proof. Let $p : \operatorname{Spec} \mathbf{C} \rightarrow B$ be a closed point of B . By assumption X_p is a cellular variety and hence has finitely generated Chow groups. The first statement then follows from theorem 3.5 and theorems 1.2 and 1.3. Let's now focus on the cases when $\dim B$ is either 2 or 3. It is a consequence of Mumford's theorem that a connected smooth projective complex variety with finitely generated Chow group of zero-cycles actually has Chow group of zero-cycles generated by a point. Thus the second and third statements follow from theorem 3.6 with $l = 1$, and from theorem 1.5 and theorem 1.4 respectively. \square

References

- [1] Donu Arapura. Motivation for Hodge cycles. *Adv. Math.*, 207(2):762–781, 2006.
- [2] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105(5):1235–1253, 1983.
- [3] Hélène Esnault, Marc Levine, and Eckart Viehweg. Chow groups of projective varieties of very small degree. *Duke Math. J.*, 87(1):29–58, 1997.
- [4] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [5] B. Brent Gordon, Masaki Hanamura, and Jacob P. Murre. Relative Chow-Künneth projectors for modular varieties. *J. Reine Angew. Math.*, 558:1–14, 2003.
- [6] André Hirschowitz and Jaya N. N. Iyer. Hilbert schemes of fat r -planes and the triviality of Chow groups of complete intersections. In *Vector bundles and complex geometry*, volume 522 of *Contemp. Math.*, pages 53–70. Amer. Math. Soc., Providence, RI, 2010.
- [7] Jaya N. N. Iyer. Chow-Künneth decomposition for a rational homogeneous bundle over a variety. Preprint.

- [8] Jaya N. N. Iyer. Murre's conjectures and explicit Chow-Künneth projections for varieties with a NEF tangent bundle. *Trans. Amer. Math. Soc.*, 361(3):1667–1681, 2009.
- [9] Uwe Jannsen. Motivic sheaves and filtrations on Chow groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 245–302. Amer. Math. Soc., Providence, RI, 1994.
- [10] Shun-Ichi Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.
- [11] Robert Laterveer. Algebraic varieties with small Chow groups. *J. Math. Kyoto Univ.*, 38(4):673–694, 1998.
- [12] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.
- [13] Anna Otwinowska. Remarques sur les groupes de Chow des hypersurfaces de petit degré. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(1):51–56, 1999.
- [14] Kapil H. Paranjape. Cohomological and cycle-theoretic connectivity. *Ann. of Math. (2)*, 139(3):641–660, 1994.
- [15] Chad Schoen. On Hodge structures and nonrepresentability of Chow groups. *Compositio Math.*, 88(3):285–316, 1993.
- [16] Charles Vial. Chow groups of smooth varieties fibred by quadrics. Preprint.
- [17] Charles Vial. Chow-Künneth decomposition for families of varieties with trivial Chow group of zero-cycles. Preprint.
- [18] Charles Vial. Niveau and coniveau filtrations on cohomology groups and Chow groups. Preprint.
- [19] Charles Vial. On the motive of fourfolds with a nef tangent bundle. Preprint.
- [20] Charles Vial. Projectors on the intermediate algebraic Jacobians. Preprint.
- [21] Charles Vial. Pure motives with representable Chow groups. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1191–1195, 2010.

DPMMS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE, CB3 0WB, UK
e-mail : `c.vial@dpmms.cam.ac.uk`